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A comprehensive extension of optimal ordering policy for stock-dependent demand under progressive payment scheme

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ABSTRACT

In a recent paper, Soni and Shah (2008) presented an inventory model with a stock-dependent demand under progressive payment scheme, assuming zero ending-inventory and adopting a cost-minimization objective. However, with a stock-dependent demand a non-zero ending stock may increase profits resulting from the increased demand. This work is motivated by Soni and Shah's (2008) paper extending their model to allow for: (1) a non-zero ending-inventory, (2) a profit-maximization objective, (3) a limited inventory capacity and (4) deteriorating items with a constant deterioration rate. For the resulted model sufficient conditions for the existence and uniqueness of the optimal solution are provided. Finally, several economic interpretations of the theoretical results are also given.

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1. Introduction

An increase in shelf space for an item induces more consumers to buy it. This occurs because of its visibility, popularity or variety. Conversely, low stocks of certain goods (e.g., baked goods) might raise the perception that they are not fresh. Therefore, building up inventory often has a positive impact on the sales, as well as the profit. Levin et al. (1972) observed that “large piles of consumer goods displayed in a supermarket will lead the customer to buy more. Yet, too much pileup in everyone's way leaves a negative impression on buyer and employee alike”. Silver and Peterson (1982) also noted that sales at the retail level tend to be proportional to the amount of inventory displayed. In order to quantify this, Baker and Urban (1988) established an economic order quantity (EOQ) model for a power-form inventory-level-dependent demand pattern (i.e., the demand rate at time t is $D(t) = \alpha[I(t)]^\beta$, where $I(t)$ is the inventory level, $\alpha > 0$, and $0 < \beta < 1$). Mandal and Phaujder (1989) then developed a production inventory model for deteriorating items with uniform rate of production and linearly stock-dependent demand (i.e., $D(t) = \alpha + \beta I(t)$, where both α and $\beta > 0$). Recently, two closely related papers were Chang et al. (2010) and Yang et al. (2010).

In practice, suppliers frequently offer retailers many incentives such as a permissible delay in payments to attract new customers and increase sales. Goyal (1985) first established an economic

order quantity (EOQ) model when a supplier offers its retailer a permissible delay in payments. Recently, Soni and Shah (2008) established an inventory model with a stock-dependent demand under progressive payment scheme. Their analysis imposed a terminal condition of zero ending-inventory and also they adopted a cost-minimization objective. However, with a stock-dependent demand, “it may be desirable to order large quantities, resulting in stock remaining at the end of the cycle, due to the potential profits resulting from the increased demand” as stated in Urban (1992). Therefore, in this paper we extend their model to allow for: (1) an ending-inventory to be non-zero, (2) a profit-maximization objective, (3) a maximum inventory ceiling to reflect either a limited shelf space or “too much piled up in everyone's way leaves a negative impression on buyer and employee alike” as stated in Levin et al. (1972), and (4) deteriorating items. In addition, we prove the solution uniquely exists. Furthermore, we provide several economic interpretations for the reader to understand the implication of our theoretical results.

2. Assumptions and notation

The assumptions of Soni and Shah (2008) are used in the model development. However, we need the following additional assumptions to extend their model.

1. The maximum inventory level is U .
2. The ending inventory level (or shelf space) could be greater than zero.

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3. The demand rate, $D(t)$, is a function of inventory-level $I(t)$ as follows:

$$D(t) = a + bI(t), \quad 0 \leq t \leq T, \quad \text{where } a, b > 0. \quad (1)$$

Some additional notation is used throughout this paper:

- h the inventory holding cost/unit/year, excluding interest charges
- P the selling price/unit
- c the unit purchase cost, with $c < P$
- A the ordering cost/order
- θ the constant deterioration rate
- T the replenishment cycle time in years (a decision variable)
- q the inventory level at time T (a decision variable)
- Q the inventory level at time 0

3. Mathematical models

An order of $Q-q$ units arrives at time $t=0$. Since the ending inventory is q units, the inventory level at time $t=0$ is Q units. Then the inventory gradually depletes to q (with $0 \leq q$) at time $t=T$ due to deterioration and consumption. Likewise, another order of $Q-q$ units is placed and delivered at time $t=T$. Thus, the replenishment cycle is repeated again. The supplier provides the retailer a progressive payment scheme. Therefore, if the retailer pays the supplier by M , then the supplier does not charge the retailer any interest. Hence, from time 0 to M , the retailer can deposit its sales revenue into an interest bearing account at the rate of le /unit/year. At time M , the retailer pays all items sold, and starts paying interest on unsold items. If the retailer pays after M but before $N(N > M)$, then the supplier charges the retailer an interest rate of lc_1 /unit/year. If the retailer pays after N , then the supplier charges the retailer an interest rate of lc_2 /unit/year ($lc_2 > lc_1$). From the assumptions, the inventory level $I(t)$ at time $t \in [0, T]$ satisfies the following differential equation:

$$\frac{dI(t)}{dt} + \theta I(t) = -a - bI(t), \quad (2)$$

with the boundary condition $I(T) = q \geq 0$. Solving (2), we get:

$$I(t) = \left(q + \frac{a}{w}\right)e^{w(T-t)} - \frac{a}{w}, \quad 0 \leq t \leq T, \quad \text{where } w = b + \theta. \quad (3)$$

Since $I(0) = Q$, we obtain

$$Q = \left(q + \frac{a}{w}\right)e^{wT} - \frac{a}{w} \leq U. \quad (4)$$

Notice that the original concept of supplier's progressive interest scheme came from Goyal et al. (2007). Recently, Soni and Shah (2009) also proposed a power-form stock-dependent demand rate (i.e., $D(t) = \alpha I(t)^\beta$, where $0 \leq t \leq T$, $\alpha > 0$, and $0 \leq \beta < 1$), instead of a linear-form stock-dependent demand rate here (i.e., $D(t) = a + bI(t)$, where $0 \leq t \leq T$, and $a, b > 0$). In a linear-form stock-dependent demand, if building up inventory is profitable, then the optimal inventory level should increase without bound until some capacity constraint is reached. Hence, we impose the maximum inventory level U here. By contrast, with a power-form stock-dependent demand, there is diminishing return in demand with respect to inventory level and the optimal inventory level should not be increased without bound. However, for both stock-dependent demand functions, the annual total profit increases if we relax the condition of a zero ending inventory as shown in Examples 1–3.

The retailer's replenishment cycle time, T , has the following three alternatives: (A) $T \leq M$, (B) $M < T < N$, and (C) $T \geq N$. We then calculate the annual total profit for each alternative accordingly.

3.1. $T \leq M$

In this case, based on profit = revenue + interest earned – purchasing cost – holding cost – ordering cost, we have the retailer's annual total profit as:

$$\Pi_1(T, q) = \frac{1}{T} \left[P \int_0^T D(t) dt + Ple \left[\int_0^T \int_0^t D(x) dx dt + (M - T) \int_0^T D(t) dt \right] - c(I(0) - q) - h \int_0^T I(t) dt - A \right].$$

After some algebraic manipulations, we can simplify it to:

$$\Pi_1(T, q) = P(1 + leM)a \left(1 - \frac{b}{w}\right) + \frac{ha}{w} + Ple \frac{b}{w} \left(q + \frac{a}{w}\right) - Ple \frac{a}{2} \left(1 - \frac{b}{w}\right) T + X \left(q + \frac{a}{w}\right) \frac{1}{w} (e^{wT} - 1) \frac{1}{T} - \frac{A}{T}, \quad (5)$$

where

$$X = Pb \left(1 + leM - \frac{le}{w}\right) - cw - h = (P - c)b + PbleM - h - c\theta - Pble \frac{1}{w}. \quad (6)$$

3.2. $M < T < N$

Similarly, we get the retailer's annual total profit as

$$\Pi_2(T, q) = \frac{1}{T} \left[P \int_0^T D(t) dt + Ple \int_0^M \int_0^t D(x) dx dt - c(I(0) - q) - h \int_0^T I(t) dt - A - clc_1 \int_M^T I(t) dt \right].$$

Taking some algebraic manipulations, we derive:

$$\Pi_2(T, q) = Pa \left(1 - \frac{b}{w}\right) + \frac{ha}{w} + clc_1 \frac{a}{w} + \left[X + \left(Ple \frac{b}{w} - clc_1\right) e^{-wM} \right] \times \left(q + \frac{a}{w}\right) \frac{1}{w} e^{wT} \frac{1}{T} + \left[\left(Ple \left(1 - \frac{b}{w}\right)\right) \frac{aM^2}{2} - clc_1 \frac{aM}{w} - A - (Pb - cw - h - clc_1) \left(q + \frac{a}{w}\right) \frac{1}{w} \right] \frac{1}{T}. \quad (7)$$

3.3. $T \geq N$

It is clear that the retailer's annual total profit is

$$\Pi_3(T, q) = \frac{1}{T} \left[P \int_0^T D(t) dt + Ple \int_0^M \int_0^t D(x) dx dt - c(I(0) - q) - h \int_0^T I(t) dt - A - clc_1 \int_M^N I(t) dt - clc_2 \int_N^T I(t) dt \right].$$

After some algebraic manipulations, it can be reduced to:

$$\Pi_3(T, q) = Pa \left(1 - \frac{b}{w}\right) + \frac{ha}{w} + clc_2 \frac{a}{w} + \left[X + \left(Ple \frac{b}{w} - clc_1\right) e^{-wM} + c \left(Ic_1 - Ic_2\right) e^{-wN} \right] \left(q + \frac{a}{w}\right) \frac{1}{w} e^{wT} \frac{1}{T} + \left[\left(Ple \left(1 - \frac{b}{w}\right)\right) \frac{aM^2}{2} - A - (Pb - cw - h - clc_2) \times \left(q + \frac{a}{w}\right) \frac{1}{w} + c(Ic_1 - Ic_2) \frac{a}{w} N - clc_1 \frac{a}{w} M \right] \frac{1}{T}. \quad (8)$$

Obviously the retailer's annual total profit function is as follows:

$$\Pi(T, q) = \begin{cases} \Pi_1(T, q), & 0 < T \leq M, \\ \Pi_2(T, q), & M < T < N, \\ \Pi_3(T, q), & T \geq N. \end{cases}$$

By substituting $T = M$ into the profit functions $\Pi_1(T, q)$ and $\Pi_2(T, q)$, we have $\Pi_1(M, q) = \Pi_2(M, q)$. Similarly, we get $\Pi_2(N, q) = \Pi_3(N, q)$.

4. Optimal solution

To find the optimal solution for the problem, we maximize $\Pi_1(T, q)$, $\Pi_2(T, q)$ and $\Pi_3(T, q)$ separately, and then compare them to obtain the maximum value.

4.1. Maximizing $\Pi_1(T, q)$

For the first subsection, the problem to be solved is:

$$\begin{aligned} \max_{(T, q)} \quad & \Pi_1(T, q) \\ \text{subject to} \quad & 0 < T \leq M, \\ & (q + \frac{a}{w})e^{wT} - \frac{a}{w} \leq U, \text{ and} \\ & q \geq 0. \end{aligned}$$

Taking the first- and the second-order derivatives of $\Pi_1(T, q)$ with respect to T and q , we obtain:

$$\frac{\partial \Pi_1(T, q)}{\partial T} = \frac{1}{T^2} \left[A - X \left(q + \frac{a}{w} \right) \frac{1}{w} (e^{wT} - 1) \right] + \frac{1}{T} X \left(q + \frac{a}{w} \right) e^{wT} - Ple \frac{a}{2} \left(1 - \frac{b}{w} \right), \quad (9)$$

$$\frac{\partial^2 \Pi_1(T, q)}{\partial T^2} = \frac{-2}{T^3} \left[A - X \left(q + \frac{a}{w} \right) \frac{1}{w} (e^{wT} - 1) \right] - \frac{1}{T^2} (2 - wT) X \left(q + \frac{a}{w} \right) e^{wT}, \quad (10)$$

$$\frac{\partial \Pi_1(T, q)}{\partial q} = Ple \frac{b}{w} + X \frac{1}{w} (e^{wT} - 1) \frac{1}{T}, \quad (11)$$

and

$$\frac{\partial^2 \Pi_1(T, q)}{\partial q^2} = 0. \quad (12)$$

Consequently, the Hessian matrix of $\Pi_1(T, q)$ is

$$\begin{pmatrix} \frac{\partial^2 \Pi_1}{\partial T^2} & \frac{\partial^2 \Pi_1}{\partial T \partial q} \\ \frac{\partial^2 \Pi_1}{\partial q \partial T} & \frac{\partial^2 \Pi_1}{\partial q^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 \Pi_1}{\partial T^2} & \frac{\partial^2 \Pi_1}{\partial T \partial q} \\ \frac{\partial^2 \Pi_1}{\partial q \partial T} & 0 \end{pmatrix} = - \left(\frac{\partial \Pi_1}{\partial T \partial q} \right)^2 < 0. \quad (13)$$

Since the Hessian matrix is negative, we cannot find an optimal inner-point solution (T^*, q^*) . In fact, $\Pi_1(T, q)$ is a continuous and differentiable function on a compact set, hence it has a global maximum. Consequently, the maximum profit must be at the boundary of the feasible region and the following cases should be examined: (1) $T = 0$, (2) $Q = U$ (i.e., Case A below), (3) $q = 0$ (i.e., Case B below) and (4) $T = \min \{M, \frac{1}{w} \ln(\frac{wU}{a} + 1)\} \equiv B_1$. It is obvious that the profit is zero for the first case. Therefore, we do not need to discuss this case. While the last case follows because T should satisfy both the constraints $T \leq M$ and $(q + \frac{a}{w})e^{wT} - \frac{a}{w} \leq U$ at $q = 0$. Then we will prove that under specific conditions only one of the Cases A and B should be examined. From $\frac{\partial \Pi_1(T, q)}{\partial q} = Ple \frac{b}{w} + X \frac{1}{w} (e^{wT} - 1) \frac{1}{T}$ as in (11), if $X \geq 0$ then $\frac{\partial \Pi_1(T, q)}{\partial q} > 0$ for every T so that $\Pi_1(T, q)$ is maximized at $Q = U$ and the optimal solution is given by Case A. If $X < 0$ then there are values for T so that $\frac{\partial \Pi_1(T, q)}{\partial q} < 0$ and values for T so that $\frac{\partial \Pi_1(T, q)}{\partial q} > 0$. We can easily confirm that: if $\frac{\partial \Pi_1(B_1, q)}{\partial q} = Ple \frac{b}{w} + X \frac{1}{w} (e^{wB_1} - 1) \frac{1}{B_1} > 0$ then the optimal solution is given by

Case A. Otherwise (if $\frac{\partial \Pi_1(B_1, q)}{\partial q} = Ple \frac{b}{w} + X \frac{1}{w} (e^{wB_1} - 1) \frac{1}{B_1} < 0$), the optimal solution is given by Case B. Note that if $B_1 = \frac{1}{w} \ln(\frac{wU}{a} + 1)$ then the candidate maximizer (i.e., $q = 0, T = \frac{1}{w} \ln(\frac{wU}{a} + 1)$) follows either from Case A or B as both the constraints $q = 0$ and $Q = U$ hold. Summarizing, if $\partial \Pi_1(T, q)/\partial q > 0$, then $\Pi_1(T, q)$ is a strictly increasing function of q . To make q as large as possible, we set $Q = U$ because of $q < Q \leq U$. Otherwise, if $\partial \Pi_1(T, q)/\partial q \leq 0$, then $\Pi_1(T, q)$ is a decreasing function of q , and is maximized at $q = 0$. As a result, only the cases (A) $Q = U$, (B) $q = 0$ should be examined, which also are mutually exclusive.

In addition, by using Taylor's series approximation to the right-hand side of (11), we obtain the following result:

$$\begin{aligned} \frac{\partial \Pi_1(T, q)}{\partial q} &= Ple \frac{b}{w} + X \frac{1}{w} (e^{wT} - 1) \frac{1}{T} \\ &\approx (P - c)b + PbleM - h - c\theta, \end{aligned} \quad (14)$$

A simple economic interpretation of (14) is as follows. For building up an additional unit of inventory, $(P - c)b$ is the profit from sales, $PbleM$ represents the benefit from supplier's trade credit, h is the inventory holding cost excluding interest charges, and $c\theta$ represents the deterioration cost. As a result, if $\partial \Pi_1(T, q)/\partial q > 0$, then building up inventory is profitable. Otherwise (i.e., $\partial \Pi_1(T, q)/\partial q \leq 0$), building up inventory is not profitable.

Case A. $Q = U$

By setting $Q = U$ in (4), we can solve for q .

$$q = Ue^{-wT} - \frac{a}{w} (1 - e^{-wT}). \quad (15)$$

Substituting (15) into $\Pi_1(T, q)$, we reduce the buyer's annual total profit function to a single decision variable T :

$$\begin{aligned} \Pi_1(T) &= P(1 + leM)a \left(1 - \frac{b}{w} \right) + \frac{ha}{w} + Ple \frac{b}{w} \left(U + \frac{a}{w} \right) e^{-wT} \\ &\quad - Ple \frac{a}{2} \left(1 - \frac{b}{w} \right) T + X \left(U + \frac{a}{w} \right) \frac{1}{w} (1 - e^{-wT}) \frac{1}{T} - \frac{A}{T}. \end{aligned} \quad (16)$$

The first-order condition for a maximum is:

$$\frac{d\Pi_1(T)}{dT} = \frac{e^{-wT}}{T^2} \left[-Pleb \left(U + \frac{a}{w} \right) T^2 - Ple \frac{a}{2} \left(1 - \frac{b}{w} \right) T^2 e^{wT} + (A - X \left(U + \frac{a}{w} \right) \frac{1}{w} e^{wT} + X \left(U + \frac{a}{w} \right) \frac{1}{w} + X \left(U + \frac{a}{w} \right) T \right] = 0 \quad (17)$$

Theorem 1. Let $T = T_{1,1}$ be the solution of (17).

(a) Eq. (17) has a unique solution.

(b) If $(P - c)b - \frac{Pble}{w} - h - c\theta > 0$ (i.e., $\partial \Pi_1(T, q)/\partial q > 0$) and $T_{1,1} \leq M$, then $T_{1,1}$ is the global maximum point of $\Pi_1(T)$.

Proof. See Appendix A. \square

If $T_{1,1} \leq M$, then we set $T_{1,1}^* = T_{1,1}$. Substituting $T_{1,1}$ into (15) we have

$$q_{1,1} = Ue^{-wT_{1,1}} - \frac{a}{w} (1 - e^{-wT_{1,1}}) \text{ and set } q_{1,1}^* = q_{1,1}. \quad (18)$$

Otherwise, we set $T_{1,1}^* = M$, and

$$q_{1,1}^* = Ue^{-wM} - \frac{a}{w} (1 - e^{-wM}). \quad (19)$$

Consequently, $(T_{1,1}^*, q_{1,1}^*)$ is the optimal solution to Case A. A simple economic interpretation of Theorem 1 is as follows. If $\partial \Pi_1(T, q)/\partial q > 0$, then building up inventory is profitable. Therefore, we should display stocks to the maximum allowable amount of U units in a supermarket without leaving a negative impression on buyers.

Case B. $q = 0$

Substituting $q = 0$ into (5), we get

$$\begin{aligned} \Pi_1(T) = & P(1 + leM)a\left(1 - \frac{b}{w}\right) + \frac{ha}{w} + Ple\frac{ab}{w^2} \\ & - Ple\frac{a}{2}\left(1 - \frac{b}{w}\right)T + X\frac{a}{w^2}(e^{wT}-1)\frac{1}{T} - \frac{A}{T}. \end{aligned} \quad (20)$$

The first-order condition for a maximum is:

$$\frac{\partial \Pi_1(T)}{\partial T} = \frac{1}{T^2} \left[A - X\frac{a}{w^2}(e^{wT}-1) + X\frac{a}{w}e^{wT}T - Ple\frac{a}{2}\left(1 - \frac{b}{w}\right)T^2 \right] = 0. \quad (21)$$

Theorem 2. Let $T = T_{1,2}$ be the solution of (21).

- (a) Eq. (21) has a unique solution.
- (b) If $X < 0$ (i.e., $\partial \Pi_1(T, q)/\partial q \leq 0$) and $T_{1,2} \leq B_1$, then $T_{1,2}$ is the global maximum point of $\Pi_1(T)$.

Proof. See Appendix B. \square

If $T_{1,2} \leq B_1$ then we set $T_{1,2}^* = T_{1,2}$. Otherwise, we set $T_{1,2} = B_1$. Consequently, $(T_{1,2}^*, 0)$ is the optimal solution to Case B. A simple economic interpretation of Theorem 2 is as follows. If $\partial \Pi_1(T, q)/\partial q \leq 0$, then building up inventory is not profitable. Therefore, we should not keep the ending inventory positive, which implies that $q = 0$. We then discuss the next subsection, in which the maximization of $\Pi_2(T, q)$ is necessary only if $B_1 = M$.

4.2. Maximizing $\Pi_2(T, q)$

For the second subsection, the problem to be solved is:

$$\begin{aligned} \max_{(T, q)} \quad & \Pi_2(T, q) \\ \text{subject to} \quad & M < T \leq N, \\ & \left(q + \frac{a}{w}\right)e^{wT} - \frac{a}{w} \leq U, \text{ and} \\ & q \geq 0. \end{aligned}$$

Taking the first- and the second-order derivatives of $\Pi_2(T, q)$ with respect to T and q , we obtain:

$$\begin{aligned} \frac{\partial \Pi_2(T, q)}{\partial T} = & \frac{1}{T^2} \left[X + \left(Ple\frac{b}{w} - clc_1 \right) e^{-wM} \right] \\ & \times \left(q + \frac{a}{w} \right) \frac{1}{w} (wT - 1) e^{wT} \\ & - \frac{1}{T^2} \left[\left(Ple\left(1 - \frac{b}{w}\right) \right) \frac{aM^2}{2} - clc_1 \frac{aM}{w} - A \right. \\ & \left. - (Pb - cw - h - clc_1) \left(q + \frac{a}{w} \right) \frac{1}{w} \right], \end{aligned} \quad (22)$$

$$\begin{aligned} \frac{\partial^2 \Pi_2(T, q)}{\partial T^2} = & \frac{1}{T^3} \left[Ple\frac{b}{w}(Mw - 1) + \left(Ple\frac{b}{w} - clc_1 \right) e^{-wM} \right. \\ & + \left(Pb - cw - h \right) \left(q + \frac{a}{w} \right) \frac{1}{w} (w^2T^2 - 2wT + 2) e^{wT} \Big] \\ & + 2\frac{1}{T^3} \left[\left(Ple\left(1 - \frac{b}{w}\right) \right) \frac{aM^2}{2} - clc_1 \frac{aM}{w} \right. \\ & \left. - A - (Pb - cw - h - clc_1) \left(q + \frac{a}{w} \right) \frac{1}{w} \right], \end{aligned} \quad (23)$$

$$\begin{aligned} \frac{\partial \Pi_2(T, q)}{\partial q} = & \left[X + \left(Ple\frac{b}{w} - clc_1 \right) e^{-wM} \right] \frac{1}{w} e^{wT} \frac{1}{T} - (Pb - cw - h - clc_1) \frac{1}{wT} \\ = & \left\{ X(e^{wT} - 1) + PleM + \left(Ple\frac{b}{w} - clc_1 \right) (e^{w(T-M)} - 1) \right\} \frac{1}{wT} \end{aligned} \quad (24)$$

and

$$\frac{\partial^2 \Pi_2(T, q)}{\partial q^2} = 0. \quad (25)$$

As in Section 4.1, the Hessian matrix of $\Pi_2(T, q)$ is negative and hence we search for a maximum on the boundary, which is either $Q = U$ (i.e., $\partial \Pi_2(T, q)/\partial q > 0$) or $q = 0$ (i.e., $\partial \Pi_2(T, q)/\partial q \leq 0$). Let us discuss them separately below.

Case A. $Q = U$

From Section 4.1, we get q as in (15). Substituting (15) into (6), we simplify $\Pi_2(T, q)$ into a single decision variable function of T as follows:

$$\Pi_2(T) = Pa\left(1 - \frac{b}{w}\right) + \frac{ha}{w} + clc_1 \frac{a}{w} + A_1 \frac{1}{T} - A_2 e^{-wT} \frac{1}{T}, \quad (26)$$

where

$$\begin{aligned} A_1 = & \left[\left(X + \left(Ple\frac{b}{w} - clc_1 \right) e^{-wM} \right) \left(U + \frac{a}{w} \right) \frac{1}{w} \right. \\ & \left. + Ple\left(1 - \frac{b}{w}\right) \frac{aM^2}{2} - clc_1 \frac{aM}{w} - A \right], \end{aligned} \quad (27)$$

and

$$A_2 = (Pb - cw - h - clc_1) \left(U + \frac{a}{w} \right) \frac{1}{w}. \quad (28)$$

The first-order condition for a maximum is:

$$\frac{\partial \Pi_2(T)}{\partial T} = \frac{1}{T^2} [A_2(wT + 1)e^{-wT} - A_1] = 0. \quad (29)$$

Theorem 3. Let $T_{2,1}$ be the solution of (29). If $A_2 > A_1 > 0$, then we have:

- (a) Eq. (29) has a unique solution.
- (b) If $M < T_{2,1} < N$ then $T_{1,2}$ is the global maximum point of $\Pi_2(T)$.

Proof. See Appendix C. \square

If $M < T_{2,1} < N$, we set $T_{2,1}^* = T_{2,1}$, and then by substituting $T_{2,1}$ into (15) we get $q_{2,1}^* = q(T_{2,1})$. If $T_{2,1} < M$, then we set $T_{2,1}^* = M$ and get $q_{2,1}^* = q(M)$ from (15). If $T_{2,1} > N$, then we set $T_{2,1}^* = N$ and get $q_{2,1}^* = q(N)$ from (15). If the conditions for a maximum do not hold, then we set

$$T_{2,1}^* = \begin{cases} M & \text{if } \Pi_2(M, q(M)) > \Pi_2(N, q(N)), \\ N & \text{if } \Pi_2(M, q(M)) \leq \Pi_2(N, q(N)) \end{cases}$$

and then obtain $q_{2,1}^*$ accordingly. Consequently, $(T_{2,1}^*, q_{2,1}^*)$ is the optimal solution to $\Pi_2(T, q)$ for the case in which $Q = U$.

Corollary 1. If (29) has a solution, then this is the global maximum point of $\Pi_2(T)$ if $X > 0$ and $Ple\frac{b}{w} - clc_1 > 0$.

Proof. It is trivial that $X > 0$ and $Ple\frac{b}{w} - clc_1 > 0$ imply $A_2 > 0$. \square

Case B. $q = 0$

Substituting $q = 0$ into (6), we obtain

$$\Pi_2(T) = Pa\left(1 - \frac{b}{w}\right) + \frac{ha}{w} + clc_1 \frac{a}{w} + A_3 e^{wT} \frac{1}{T} + A_4 \frac{1}{T}, \quad (30)$$

where

$$A_3 = \left(X + \left(Ple \frac{b}{w} - clc_1\right) e^{-wM}\right) \frac{a}{w^2}, \quad (31)$$

and

$$A_4 = \left(Ple\left(1 - \frac{b}{w}\right)\right) \frac{aM^2}{2} - clc_1 \frac{aM}{w} - A - (Pb - cw - h - clc_1) \frac{a}{w^2}. \quad (32)$$

The first-order condition for a maximum is:

$$\frac{\partial \Pi_2(T)}{\partial T} = \frac{1}{T^2} [A_3(wT - 1)e^{wT} - A_4] = 0. \quad (33)$$

Theorem 4. Let $T_{2,2}$ be the solution of (33). If $A_4 < A_3 < 0$, then we get:

- (a) Eq. (33) has a unique solution.
- (b) If $M < T_{2,2} < B_2$, where $B_2 = \min\{N, \frac{1}{w} \ln(\frac{wU}{a} + 1)\}$ then $T_{2,2}$ is the global maximum point of $\Pi_2(T)$.

Proof. See Appendix D. \square

If $M < T_{2,2} < B_2$, then we set $T_{2,2}^* = T_{2,2}$. If $T_{2,2} \leq M$, then we set $T_{2,2}^* = M$. If $T_{2,2} \geq B_2$, then we set $T_{2,2}^* = B_2$. If the conditions for a maximum do not hold, then we set

$$T_{2,2}^* = \begin{cases} M & \text{if } \Pi_2(M, 0) > \Pi_2(B_2, 0), \\ B_2 & \text{if } \Pi_2(M, 0) \leq \Pi_2(B_2, 0). \end{cases}$$

Consequently, $(T_{2,2}^*, 0)$ is the optimal solution to $\Pi_2(T, q)$ for the case in which $q = 0$.

Corollary 2. If (33) has a solution, then this is the global maximum point of $\Pi_2(T)$ if $X < 0$ and $Ple \frac{b}{w} - clc_1 < 0$.

Proof. It is clear that $X < 0$ and $Ple \frac{b}{w} - clc_1 < 0$ imply $A_3 < 0$. \square

We then discuss the last subsection, in which the maximization of $\Pi_3(T, q)$ is necessary only if $B_2 = N$.

4.3. Maximizing $\Pi_3(T, q)$

For the third and last subsection, the problem to be solved is:

$$\begin{aligned} & \max_{(T, q)} \quad \Pi_3(T, q) \\ & \text{subject to} \quad N \leq T, \\ & \quad \quad \quad \left(q + \frac{a}{w}\right) e^{wT} - \frac{a}{w} \leq U, \text{ and} \\ & \quad \quad \quad q \geq 0. \end{aligned}$$

Hence, the resulted feasible region is a compact set.

The first- and the second-order derivatives of $\Pi_3(T, q)$ with respect to T and q are:

$$\begin{aligned} \frac{\partial \Pi_3(T, q)}{\partial T} &= \frac{1}{T^2} \left[X + \left(Ple \frac{b}{w} - clc_1 \right) e^{-wM} + c(Ic_1 - Ic_2) e^{-wN} \right] \\ & \quad \times \left(q + \frac{a}{w} \right) \frac{1}{w} (wT - 1) e^{wT} - \frac{1}{T^2} \left[\left(Ple \left(1 - \frac{b}{w} \right) \right) \right. \\ & \quad \times \frac{aM^2}{2} - A - (Pb - cw - h - clc_2) \left(q + \frac{a}{w} \right) \frac{1}{w} \\ & \quad \left. + c(Ic_1 - Ic_2) \frac{a}{w} N - clc_1 \frac{a}{w} M \right], \quad (34) \end{aligned}$$

$$\begin{aligned} \frac{\partial \Pi_3(T, q)}{\partial q} &= \left[X + \left(Ple \frac{b}{w} - clc_1 \right) e^{-wM} + c(Ic_1 - Ic_2) e^{-wN} \right] \\ & \quad \times \frac{1}{w} e^{wT} \frac{1}{T} - (Pb - cw - h - clc_2) \frac{1}{w} \frac{1}{T} \\ &= \left\{ X(e^{wT} - 1) + PbleM + \left(Ple \frac{b}{w} - clc_1 \right) (e^{w(T-M)} - 1) \right. \\ & \quad \left. + (clc_1 - clc_2) (e^{w(T-N)} - 1) \right\} \frac{1}{wT}, \quad (35) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \Pi_3(T, q)}{\partial T^2} &= \frac{1}{T^3} \left[X + \left(Ple \frac{b}{w} - clc_1 \right) e^{-wM} + c(Ic_1 - Ic_2) e^{-wN} \right] \\ & \quad \times \left(q + \frac{a}{w} \right) \frac{1}{w} (w^2 T^2 - 2wT + 2) e^{wT} \\ & \quad + \frac{2}{T^3} \left[\left(Ple \left(1 - \frac{b}{w} \right) \right) \frac{aM^2}{2} - A - (Pb - cw - h - clc_2) \right. \\ & \quad \left. \times \left(q + \frac{a}{w} \right) \frac{1}{w} + c(Ic_1 - Ic_2) \frac{a}{w} N - clc_1 \frac{a}{w} M \right] \quad (36) \end{aligned}$$

and

$$\frac{\partial^2 \Pi_3(T, q)}{\partial q^2} = 0. \quad (37)$$

As in Sections 4.1 and 4.2 the Hessian matrix is negative and hence we search for a maximum on the boundary, which is either $Q = U$ (i.e., $\partial \Pi_3(T, q)/\partial q > 0$) or $q = 0$ (i.e., $\partial \Pi_3(T, q)/\partial q \leq 0$). Again, we discuss these two cases separately.

Case A. $Q = U$

It is clear that q is the same as in (15). Substituting (15) into (7), we simplify $\Pi_3(T, q)$ into a single decision variable function of T .

$$\Pi_3(T) = Pa\left(1 - \frac{b}{w}\right) + \frac{ha}{w} + clc_2 \frac{a}{w} + A_5 \frac{1}{T} - A_6 e^{-wT} \frac{1}{T}, \quad (38)$$

where

$$\begin{aligned} A_5 &= \left(X + \left(Ple \frac{b}{w} - clc_1 \right) e^{-wM} + c(Ic_1 - Ic_2) e^{-wN} \right) \left(U + \frac{a}{w} \right) \frac{1}{w} \\ & \quad + Ple \left(1 - \frac{b}{w} \right) \frac{aM^2}{2} - A + c(Ic_1 - Ic_2) \frac{a}{w} N - clc_1 \frac{a}{w} M \end{aligned} \quad (39)$$

and

$$A_6 = (Pb - cw - h - clc_2) \left(U + \frac{a}{w} \right) \frac{1}{w}. \quad (40)$$

The first-order condition for a maximum is:

$$\frac{\partial \Pi_3(T)}{\partial T} = \frac{1}{T^2} [A_6(wT + 1)e^{-wT} - A_5] = 0. \quad (41)$$

Theorem 5. Let $T_{3,1}$ be the solution of (41). If $A_6 > A_5 > 0$, then we obtain:

- (a) Eq. (41) has a unique solution.
- (b) If $T_{3,1} > N$ then $T_{3,1}$ is the global maximum point of $\Pi_3(T)$.

Proof. The proof of this theorem is similar to that of Theorem 3 noting the similarity of Eqs. (29) and (41). Also it is worthwhile to note that $A_5 < A_1$ and $A_6 < A_2$. \square

If $T_{3,1} > N$, we set $T_{3,1}^* = T_{3,1}$, and then substitute $T_{3,1}$ into (15) to get $q_{3,1}^* = q(T_{3,1})$. If $T_{3,1} \leq N$ or if the conditions for a maximum do not hold, then we set $T_{3,1}^* = N$ and get $q_{3,1}^* = q(N)$ from (15). Consequently, $(T_{3,1}^*, q_{3,1}^*)$ is the optimal solution for the case in which $Q = U$.

Corollary 3. If (41) has a solution, then this is the global maximum point of $\Pi_3(T)$ if $X > 0$ and $Ple \frac{b}{w} - clc_2 > 0$.

Proof. It is obvious that $X > 0$ and $Ple \frac{b}{w} - clc_2 > 0$ imply $A_6 > 0$. \square

Case B. $q = 0$

Substituting $q = 0$ into (7), we have

$$\Pi_3(T) = Pa \left(1 - \frac{b}{w}\right) + \frac{ha}{w} + clc_2 \frac{a}{w} + A_7 e^{wT} \frac{1}{T} + A_8 \frac{1}{T}, \quad (42)$$

where

$$A_7 = \left[X + \left(Ple \frac{b}{w} - clc_1 \right) e^{-wM} + c(Ic_1 - Ic_2) e^{-wN} \right] \frac{a}{w^2}, \quad (43)$$

and

$$A_8 = Ple \left(1 - \frac{b}{w}\right) \frac{aM^2}{2} - A - (Pb - cw - h - clc_2) \frac{a}{w^2} + c(Ic_1 - Ic_2) \frac{a}{w} N - clc_1 \frac{a}{w} M. \quad (44)$$

The first-order condition for a maximum is:

$$\frac{\partial \Pi_3(T)}{\partial T} = \frac{1}{T^2} [A_7(wT - 1)e^{wT} - A_8] = 0. \quad (45)$$

Theorem 6. Let $T_{3,2}$ be the solution of (45). If $A_8 < A_7 < 0$, then we get:

- (a) Eq. (45) has a unique solution.
- (b) If $N \leq T_{3,2} \leq B_3$, where $B_3 = \frac{1}{w} \ln \left(\frac{wU}{a} + 1 \right)$, then $T_{3,2}$ is the global maximum point of $\Pi_3(T)$.

Proof. The proof is similar to that of Theorem 4 observing the similarity of Eqs. (33) and (45). Also it is worthwhile to note that $A_7 < A_3$ and $A_8 < A_4$. \square

If $N \leq T_{3,2} \leq B_3$, we set $T_{3,2}^* = T_{3,2}$. If $T_{3,2} \leq N$, then we set $T_{3,2}^* = N$. If $T_{3,2} \geq B_3$, then we set $T_{3,2}^* = B_3$. If the conditions for a maximum do not hold, then we set

$$T_{3,2}^* = \begin{cases} N & \text{if } \Pi_3(N, 0) > \Pi_3(B_3, 0), \\ B_3 & \text{if } \Pi_3(N, 0) \leq \Pi_3(B_3, 0). \end{cases}$$

Consequently, $(T_{3,2}^*, 0)$ is the optimal solution for the case in which $q = 0$.

Corollary 4. If (45) has a solution, then this is the global maximum point of $\Pi_3(T)$ if $X < 0$ and $Ple \frac{b}{w} - clc_2 < 0$.

Proof. It is trivial that $X < 0$ and $Ple \frac{b}{w} - clc_2 < 0$ imply $A_7 < 0$. \square

5. Numerical examples and comparisons

In Example 1, we use the same parametric values as in Soni and Shah (2008) for a linear-form stock dependent demand. However, we extend their model from non-deteriorating items (i.e., $\theta = 0$) to deteriorating items (i.e., $\theta = 0.05$). Then, in Example 2 we show that the total profit, based on the optimal solution to minimize the annual total cost in a system with a stock-dependent demand rate, is significantly less than that to maximize the total annual profit. Finally, in Example 3 we study the effect of relaxing the assumption of a zero ending inventory for a power-form stock-dependent demand.

Example 1. Let $a = 1000$, $\theta = 0.05$, $b = 3.5$, $A = 200$, $c = 20$, $P = 30$, $h = 0.2$, $le = 12\%$, $Ic_1 = 13\%$, $Ic_2 = 18\%$, $M = 17/365$, $N = 30/365$ and $U = 500$ in appropriate units. By using the above procedure, we obtain the computational results as shown in Tables 1–3. Consequently, we obtain the optimal solution as follows:

$$(T^*, q^*) = (0.06, 349.34), \text{ and } \Pi(T^*, q^*) = 20899.5.$$

If we impose $q = 0$ as in Soni and Shah (2008), then we know from Tables 1–3 with $q = 0$ that the optimal solution would be

$$(T^*, q^*) = (0.29, 0), \text{ and } \Pi(T^*, q^*) = 15925.3.$$

Now, it is obvious that to impose a zero ending inventory in an inventory system with a stock-dependent demand rate will lose profit significantly. In fact, we know from (14) that

$$\begin{aligned} (P - c)b + PbleM - h - c\theta &= (30 - 20)3.5 + (30)(3.5)(0.12) \\ &\quad \times (17/365) - 0.2 - (20)(0.05) \\ &= 34.38685 > 0. \end{aligned}$$

Hence, building up inventory is profitable, which in turn implies the optimal solution is in the case of $Q = U$.

Example 2. To compare our results with those in Soni and Shah (2008), we set $\theta = 0$ here, and let the rest of the parameters to be the same as in Example 1. Likewise, we obtain the optimal solution as follows:

$$(T^*, q^*) = (0.06, 352.27), \text{ and } \Pi(T^*, q^*) = 21343.2.$$

In contrast, their optimal solution to minimize the retailer's annual total cost is

$$(T^*, q^*) = (0.18, 0), \text{ and } \Pi(T^*, q^*) = 12654.3.$$

Again, it is clear that the retailer's annual total profit based on their optimal solution to minimize the annual total cost in a system with a stock-dependent demand rate does not induce to an optimal profit.

Example 3. We study the effect of relaxing the assumption of a zero ending inventory in a power-form stock-dependent demand as shown in Soni and Shah (2009). Let $\alpha = 1000$, $\theta = 0.05$, $\beta = 0.1$, $A = 200$, $c = 20$, $P = 30$, $h = 0.2$, $le = 12\%$, $Ic_1 = 13\%$, $Ic_2 = 18\%$, $M = 17/365$, $N = 30/365$ and $U = 500$ in appropriate units. By using the same procedure, we obtain the computational results as shown in Tables 4–6. Consequently, the optimal solution is as follows:

$$(T^*, q^*) = (0.207, 131.984) \text{ and } \Pi(T^*, q^*) = 15661.4.$$

Table 1
Case 1 ($T \leq M$).

Set $Q = U$	$Q = U = 500$	$T = M = 0.047$	$q = 380.87$	$\Pi = 20755.6$
Set $q = 0$	$Q = 50.65$	$T = M = 0.047$	$q = 0$	$\Pi = 6631.78$

Table 2
Case 2 ($M < T < N$).

Set $Q = U$	$Q = U = 500$	$T = 0.06$	$q = 349.34$	$\Pi = 20899.5$
Set $q = 0$	$Q = 95.44$	$T = N = 0.082$	$q = 0$	$\Pi = 9140.39$

Table 3
Case 3 ($T \geq N$).

Set $Q = U$	$Q = U = 500$	$T = N = 0.082$	$q = 302.18$	$\Pi = 20701$
Set $q = 0$	$Q = U = 500$	$T = 0.29$	$q = 0$	$\Pi = 15925.3$

If we impose $q = 0$ as in Soni and Shah (2009), then from Tables 4–6 (for $q = 0$) the optimal solution is:

$$(T^*, q^*) = (0.29, 0), \text{ and } \Pi(T^*, q^*) = 15268.3.$$

Again, this example indicates that to impose a zero ending inventory in an inventory system with a power-form inventory level – dependent demand function, causes a loss in profit.

With a power-form stock-dependent demand, there is diminishing return in demand with respect to inventory level. Hence, the optimal order quantity Q is not always at the maximum inventory U . To show this, we study the sensitivity analysis on the optimal order quantity Q with respect to α , as shown in Table 7. Table 7 reveals that a higher value of α causes higher values of q , Q , and Π while a lower value of T .

6. Sensitivity analysis

By using the same data as in Example 1, we study the effect of the changes in a single parameter (i.e., keeping the other parameters constant) on the optimal solution (T^*, q^*) as shown in Table 8. Table 8 reveals that (1) an increase in U or I_e causes a decrease in T^* while an increase in both q^* and $\Pi(T^*, q^*)$, (2) an increase in I_{c1} causes a decrease in both T^* and $\Pi(T^*, q^*)$ but an increase in q^* , and (3) the changes in M , N , or I_{c2} seems not to affect any of T^* , q^* , and $\Pi(T^*, q^*)$.

7. Conclusions

Recently, Soni and Shah (2008, 2009) first formulated two interesting and relevant inventory models in which the demand rate is either a linear function or a power function of the stock level under a supplier's progressive payment scheme. In this paper, we have extended their model to allow for: (1) an ending-inventory to be non-zero, (2) a profit-maximization model, (3) a maximum inventory ceiling, and (4) deteriorating items. Then we have provided the sufficient conditions for the existence and uniqueness of the optimal solution. In addition, we have explained the economic

Table 4

Case 1 ($T \leq M$).

Set $Q = U$	$Q = U = 500$	$T = M = 0.047$	$q = 413.03$	$\Pi = 13758$
Set $q = 0$	$Q = 63.56$	$T = M = 0.047$	$q = 0$	$\Pi = 9421.3$

Table 5

Case 2 ($M < T < N$).

Set $Q = U$	$Q = U = 500$	$T = N = 0.082$	$q = 347.84$	$\Pi = 15018.5$
Set $q = 0$	$Q = 119.57$	$T = N = 0.082$	$q = 0$	$\Pi = 12068.3$

Table 6

Case 3 ($T \geq N$).

Set $Q = U$	$Q = U = 500$	$T = 0.207$	$q = 131.984$	$\Pi = 15661.4$
Set $q = 0$	$Q = U = 500$	$T = 0.29$	$q = 0$	$\Pi = 15268.3$

Table 7

The optimal solution for different values of α .

α	Optimal T , q and Q	Optimal Π
100	$T = 0.96$, $q = 0.80099$, $Q = 147.608$	974.207
300	$T = 0.62$, $q = 16.8078$, $Q = 326.193$	3897.4
500	$T = 0.498$, $q = 49.34$, $Q = 484.147$	7147.18
1000	$T = 0.207$, $q = 131.984$, $Q = U = 500$	15661.4

Table 8

Sensitivity analysis.

Parameter	Percentage of changes (%)	T	q	Percentage of profit changes
U	–50	0.082(=N)	115.45	–0.35
	–25	0.07	230.05	–0.18
	+25	0.052	471.58	0.18
	+50	0.047(=M)	592.77	0.37
M	–50	0.067	333.6	–0.02
	–25	0.064	340.98	–0.01
	+25	0.059(=M)	352.28	0.01
	+50	0.067	334.82	0.02
N	–25	0.06	349.34	0
	+25	0.06	349.34	0
	+50	0.06	349.34	0
I_e	–50	0.061	346.67	–0.004
	–25	0.06	348	–0.002
	+25	0.06	350.7	0.002
	+50	0.059	352.07	0.004
I_{c1}	–50	0.065	339.09	0.006
	–25	0.063	344.01	0.003
	+25	0.058	355.13	–0.002
	+50	0.055	361.46	–0.004
I_{c2}	–50	0.058	354.67	0
	–25	0.058	354.67	0
	+25	0.058	354.67	0
	+50	0.058	354.67	0

interpretations of the theoretical results. Furthermore, we have used the same numerical example as in Soni and Shah (2008) to show that (a) by imposing on the ending inventory to be zero, or (b) by minimizing the annual total cost in an inventory model with a stock-dependent demand rate does not provide an optimal profit. Finally, we have studied the sensitivity analysis of a single parameter on the effect of the optimal solution.

While this research generalizes the inventory model by Soni and Shah (2008), further investigation can be conducted in a number of directions. For instance, we may extend the proposed model to allow for partial backlogging, poor-quality products, one time discount, and others. Also, we could consider the effects of inflation rate, defective rate, and inspection rate on the problem. Finally, we should study the supply chain coordination between the supplier and the retailer.

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Appendix A. Proof of Theorem 1

(a) From (17), we set

$$F(T) = -Pleb\left(U + \frac{a}{w}\right)T^2 + X\left(U + \frac{a}{w}\right)T + X\left(U + \frac{a}{w}\right)\frac{1}{w}, \text{ and}$$

$$G(T) = \left[Ple\frac{a}{2}\left(1 - \frac{b}{w}\right)T^2 + \left(X\left(U + \frac{a}{w}\right)\frac{1}{w} - A\right)\right]e^{wT}.$$

If there exists a unique $T > 0$ such that $F(T) = G(T)$, then (17) has a unique solution. Taking the first and second derivatives of $F(T)$, we get

$$F'(T) = -2Pleb(U + \frac{a}{w})T + X(U + \frac{a}{w}), \text{ and} \\ F''(T) = -2Pleb(U + \frac{a}{w}) < 0.$$

Hence $F(T)$ is a strictly concave function. Similarly, we have

$$G'(T) = (\gamma_1 w T^2 + 2\gamma_1 T + \gamma_2) e^{wT}, \text{ and} \\ G''(T) = (\gamma_1 w^2 T^2 + 4\gamma_1 w T + 2\gamma_1 + w\gamma_2) e^{wT}$$

where $\gamma_1 = Ple \frac{a}{2} (1 - \frac{b}{w}) > 0$ and $\gamma_2 = (X(U + \frac{a}{w}) \frac{1}{w} - A)w$.

If $0 < \gamma_1 < w\gamma_2$, then $G'(T) > 0$ ($G'(T) > 0$ if $4\gamma_1^2 - 4\gamma_1 w\gamma_2 < 0$ or equivalently $0 < \gamma_1 < w\gamma_2$). Otherwise, if $\gamma_1 \geq w\gamma_2$, then $G''(T) > 0$. Therefore, $G(T)$ is either an increasing function or a strictly convex function in T . In addition, we know

$$F(0) = X(U + \frac{a}{w}) \frac{1}{w} > G(0) = X(U + \frac{a}{w}) \frac{1}{w} - A = \frac{\gamma_2}{w}, \text{ while } F(\infty) < G(\infty).$$

Consequently, there exists a unique T such that $F(T) = G(T)$, hence $d\Pi_1(T)/dT = 0$ has a unique solution.

(b) If $T = T_{1,1}$ is the solution to $d\Pi_1(T)/dT = 0$, the second-order derivative of $\Pi_1(T)$ at this point is:

$$\left. \frac{d^2 \Pi_1(T)}{dT^2} \right|_{T=T_{1,1}} = Pleb \left(U + \frac{a}{w} \right) e^{-wT_{1,1}} \left(w - \frac{2}{T} \right) - \frac{1}{T_{1,1}} Plea \left(1 - \frac{b}{w} \right) \\ - X \left(U + \frac{a}{w} \right) w \frac{e^{-wT_{1,1}}}{T_{1,1}} < 0$$

provided that $(P - c)b - \frac{Pble}{w} - h - c\theta > 0$ which implies that $X > 0$.

The case of $Q = U$ (i.e., $\partial \Pi_1(T, q)/\partial q > 0$) and the case of $q = 0$ (i.e., $\partial \Pi_1(T, q)/\partial q \leq 0$) are mutually exclusive. Consequently, if $\partial \Pi_1(T, q)/\partial q > 0$, then the optimal solution obtained under the condition of $\partial \Pi_1(T, q)/\partial q > 0$ is clearly more profitable than the solution obtained under the condition of $\partial \Pi_1(T, q)/\partial q \leq 0$. Hence, we have proved that $T_{1,1}$ is the unique global maximum point of $\Pi_1(T, q)$.

Appendix B. Proof of Theorem 2

(a) By using (21), we let

$$F(T) = -Ple \frac{a}{2} (1 - \frac{b}{w}) T^2 + X \frac{a}{w^2} + A, \text{ and} \\ G(T) = X \frac{a}{w} e^{wT} (1 - \frac{b}{w}).$$

If there exists a unique $T > 0$ such that $F(T) = G(T)$, then (21) has a unique solution. Since $F'(T) = -Ple \frac{a}{2} (1 - \frac{b}{w}) T < 0$, $F(T)$ is a strictly decreasing function for $T > 0$.

In contrast, $G'(T) = Xae^{wT}$. If $\partial \Pi_1(T, q)/\partial q \leq 0$, then we know from (11) that $X < 0$ and thus $G(T)$ is a strictly increasing function for $T > 0$. Furthermore, $F(0) = X \frac{a}{w^2} + A > G(0) = X \frac{a}{w}$ while $F(\infty) < F(0) < G(\infty)$. Consequently, there exists a unique T such that $F(T) = G(T)$, hence $d\Pi_1(T)/dT = 0$ has a unique solution.

(b) If $T = T_{1,2}$ is the solution to $d\Pi_1(T)/dT = 0$, the second-order derivative of $\Pi_1(T)$ at this point is:

$$\left. \frac{d^2 \Pi_1(T)}{dT^2} \right|_{T=T_{1,2}} = \frac{1}{T} \left(Xae^{wT} - Plea \left(1 - \frac{b}{w} \right) \right) < 0, \text{ if } X \leq 0.$$

The case of $Q = U$ (i.e., $\partial \Pi_1(T, q)/\partial q > 0$) and the case of $q = 0$ (i.e., $\partial \Pi_1(T, q)/\partial q \leq 0$) are mutually exclusive. Consequently, if $\partial \Pi_1(T, q)/\partial q \leq 0$, then the optimal solution obtained under the condition of $\partial \Pi_1(T, q)/\partial q \leq 0$ is clearly more profitable than the solution obtained under the condition of $\partial \Pi_1(T, q)/\partial q > 0$. Hence, we have proved that is the unique global maximum point of $\Pi_1(T, q)$.

Appendix C. Proof of Theorem 3

(a) Let us set $F(T) = A_2(wT + 1)e^{-wT} - A_1$, then $d\Pi_2(T)/dT = 0 \iff F(T) = 0$. To examine if the equation $d\Pi_2(T)/dT = 0$ has a solution we examine the behavior of function $F(T)$.

$$F'(T) = -A_2 w^2 T e^{-wT} \text{ and } F(0) = A_2 - A_1.$$

So, if $A_2 > 0$ and $A_2 > A_1 > 0$, $d\Pi_2(T)/dT = 0$ has a unique solution.

(b) If $T = T_{2,1}$ is the solution to $d\Pi_2(T)/dT = 0$, the second-order derivative of $\Pi_2(T)$ at this point is:

$$\left. \frac{d^2 \Pi_2(T)}{dT^2} \right|_{T=T_{2,1}} = -\frac{A_2}{T_{2,1}} w^2 e^{-wT_{2,1}} < 0, \text{ if } A_2 > 0.$$

Hence $T_{2,1}$ is the global maximum point.

In any other case $d\Pi_2(T)/dT = 0$ has no solution or it has a solution which is not a global maximum. Since $A_1, A_2 > 0$ implies that $\partial \Pi_2(T, q)/\partial q > 0$, and $T_{2,1}$ is the unique global maximum point of $\Pi_2(T, q)$.

Appendix D. Proof of Theorem 4

(a) Let us set $F(T) = A_3(wT - 1)e^{wT} - A_4$, then from $d\Pi_2(T)/dT = 0$ we get $F(T) = 0$. To examine if the equation $d\Pi_2(T)/dT = 0$ has a solution we examine the behavior of function $F(T)$. Since $F'(T) = w^2 A_3 e^{wT}$ and $F(0) = A_3 - A_4$, if $A_3 < 0$ and $A_3 - A_4 > 0$, $d\Pi_2(T)/dT = 0$ has a unique solution. (b) If $T = T_{2,2}$ is the solution to $d\Pi_2(T)/dT = 0$, the second-order derivative of $\Pi_2(T)$ at this point is:

$$\left. \frac{d^2 \Pi_2(T)}{dT^2} \right|_{T=T_{2,2}} = \frac{w^2}{T} e^{wT} A_3 < 0, \text{ if } A_3 < 0.$$

Hence $T_{2,2}$ is the global maximum point.

Note that $d\Pi_2(T)/dT = 0$ has always at least one solution excluding the case $A_4 > -A_3$.

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